Markov and decomposability properties of the Kikuchi approximation

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Outline of the presentation

- 1. Region-based decompositions
- 2. Clique-based decompositions
- 3. Kikuchi approximation of the probability
- 4. Properties of the Kikuchi approximation
  - 4.1 Markov properties
  - 4.2 Decomposability properties
- 5. Partial Kikuchi approximations
- 6. Applications and future work

1. Region-based decompositions

- Region-based approximations of the energy
- Bethe approximation of the energy (Bethe, 1935)
- Kikuchi approximation of the energy (Kikuchi, 1951)
- Use of region-based decompositions in probabilistic inference (Yedidia, 2004)

1. Region-based decompositions

- Region R of the independence graph $G = (V, E)$
- Graph region-based decomposition $(\mathcal{R}, U)$
- Valid decomposition
- Clique-based decomposition

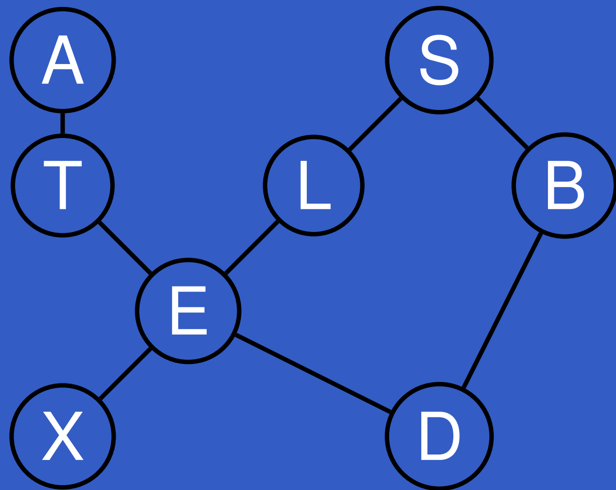
3. Kikuchi approximation

Kikuchi approximation of a probability distribution $p(\mathbf{x})$:

$$k(\mathbf{x}) = \prod_{R \in \mathcal{R}} p(\mathbf{x}_R)^{c_R}, \tag{1}$$

$$c_R = 1 - \sum_{\substack{S \in \mathcal{R} \\ R \subset S}} c_S \tag{2}$$

3. Kikuchi approximation



$$k(\mathbf{x}) = \frac{p(x_A, x_T)p(x_T, x_E)p(x_E, x_X)p(x_E, x_L)p(x_L, x_S)p(x_S, x_B)p(x_B, x_D)p(x_E, x_D)}{p(x_T)p(x_E)^3 p(x_L)p(x_S)p(x_B)p(x_D)} \tag{3}$$

4.1. Markov properties of the Kikuchi

- Pair-wise Markov property of the Kikuchi approximation is satisfied.
- Local Markov property of the Kikuchi approximation is satisfied.
- Global Markov property of the Kikuchi approximation is not satisfied.

4.1. Markov properties

Theorem 1 (Pairwise Markov property). *Given a Kikuchi approximation* $k(\mathbf{x})$ *defined on a graph* G*, and two variables* X_i *and* X_j*, if the corresponding vertices are not joined in* G*:*

$$k(x_i, x_j \mid \mathbf{x} \backslash (x_i, x_j)) = k(x_i \mid \mathbf{x} \backslash (x_i, x_j)) k(x_j \mid \mathbf{x} \backslash (x_i, x_j$$

4.1. Markov properties

Theorem 2 (Local Markov property). *Given a Kikuchi approximation $k(\mathbf{x})$ defined on a graph G, and a variable X_i,*

$$k(x_i, \mathbf{x} \backslash cl(x_i) \mid bd(x_i)) = k(x_i | bd(x_i)) k(\mathbf{x} \backslash cl(x_i) | bd(x_i))$$

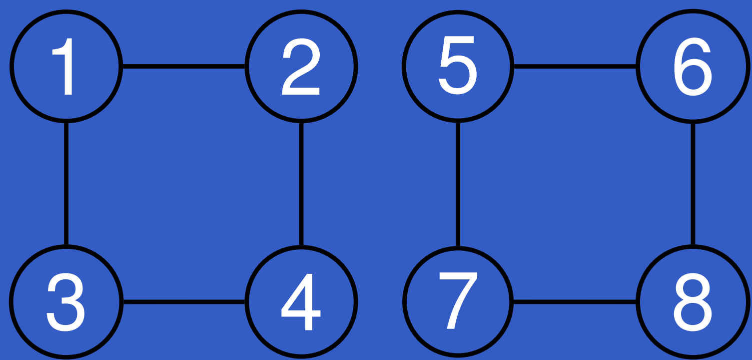
4.1. Markov properties

Conjecture 1 (Global Markov property). *Given a Kikuchi approximation* $k(\mathbf{x})$ *defined on a graph* G*, for all disjoint subsets* $\mathbf{X}_A$*,* $\mathbf{X}_B$*, and* $\mathbf{X}_C$*, whenever* $\mathbf{X}_B$ *and* $\mathbf{X}_C$ *are separated by* $\mathbf{X}_A$ *in the graph, then:*

$$k(\mathbf{x}_B, \mathbf{x}_C \mid \mathbf{x}_A) = k(\mathbf{x}_B \mid \mathbf{x}_A) k(\mathbf{x}_C \mid \mathbf{x}_A)$$

4.1. Markov properties.

Counterexample of conjecture 1. It corresponds a simple case, when $\mathbf{X}_B = X_i$, $\mathbf{X}_C = X_j$, X_i and X_j are not connected in the graph, and $\mathbf{X}_A = \emptyset$.



4.2. Decomposability properties

Theorem 3 (Kikuchi decomposition property). *Given a Kikuchi approximation* $k(\mathbf{x})$ *defined on a graph* G, *such that* $\mathbf{X} = \mathbf{X}_A \cup \mathbf{X}_B \cup \mathbf{X}_C$, *and* $\mathbf{X}_A$ *is a separator of* $\mathbf{X}_B$ *and* $\mathbf{X}_C$, *then:*

$$k(\mathbf{x}) = \frac{k_{AB}(\mathbf{x}_A, \mathbf{x}_B) k_{AC}(\mathbf{x}_A, \mathbf{x}_C)}{k_A(\mathbf{x}_A)} \tag{4}$$

4.2. Decomposability properties

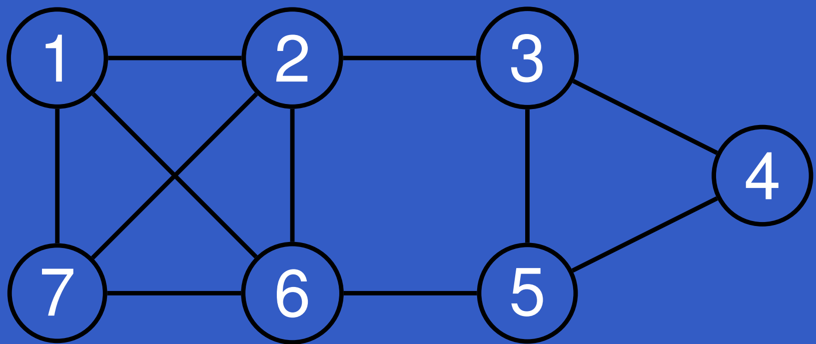
Definition 1. *There exists a decomposition of the random vector* $\mathbf{X}$ *with respect to a probability distribution* $p(\mathbf{x})$ *or equivalently,* $\mathbf{X}$ *is reducible, if and only if there exists a partition* $\mathbf{X}$ *into* $(\mathbf{X}_A, \mathbf{X}_B, \mathbf{X}_C)$ *such that:*

1. $p(\mathbf{x}_B, \mathbf{x}_C|\mathbf{x}_A) = p(\mathbf{x}_B|\mathbf{x}_A)p(\mathbf{x}_C|\mathbf{x}_A)$ *and neither* B *nor* C *are empty; and*
2. *the subgraph on* A*, in the independence graph of* $\mathbf{X}$ *is complete.*

If so, the components of $\mathbf{X}$ *are* $\mathbf{X}_{AB} = (\mathbf{X}_A, \mathbf{X}_B)$ *and* $\mathbf{X}_{AC} = (\mathbf{X}_A, \mathbf{X}_C)$*. If such a decomposition does not exist* $\mathbf{X}$ *is said to be irreducible.*

4.2. Decomposability properties

Example 1. *Consider the partition defined by sets* $A = \{2, 6\}$, $B = \{1, 7\}$ *and* $C = \{3, 4, 5\}$.



4.2. Decomposability properties of the

Theorem 4. *Given the independence graph G of $\mathbf{X}$, and the Kikuchi approximation $k(\mathbf{x})$ defined on G, if there exists a partition $\mathbf{X}$ into $(\mathbf{X}_A, \mathbf{X}_B, \mathbf{X}_C)$ such that the components of $\mathbf{X}$ are $\mathbf{X}_{AB} = (\mathbf{X}_A, \mathbf{X}_B)$ and $\mathbf{X}_{AC} = (\mathbf{X}_A, \mathbf{X}_C)$, then:*

$$k(\mathbf{x}) = \frac{k_{AB}(\mathbf{x}_{AB}) k_{AC}(\mathbf{x}_{AC})}{k_A(\mathbf{x}_A)} \quad (5)$$

5. Partial Kikuchi approximations

- Some of the irreducible incomplete components are approximated by the corresponding Kikuchi approximation.
- The rest of components are calculated exactly.

6. Applications and future work

- Definition of accuracy measures for Kikuchi approximations.
- Methods for learning Kikuchi approximations from data.
- Use of Kikuchi approximations in classification.
- Use of Kikuchi approximations in optimization.