Approximate inference and structural learning of MTE networks

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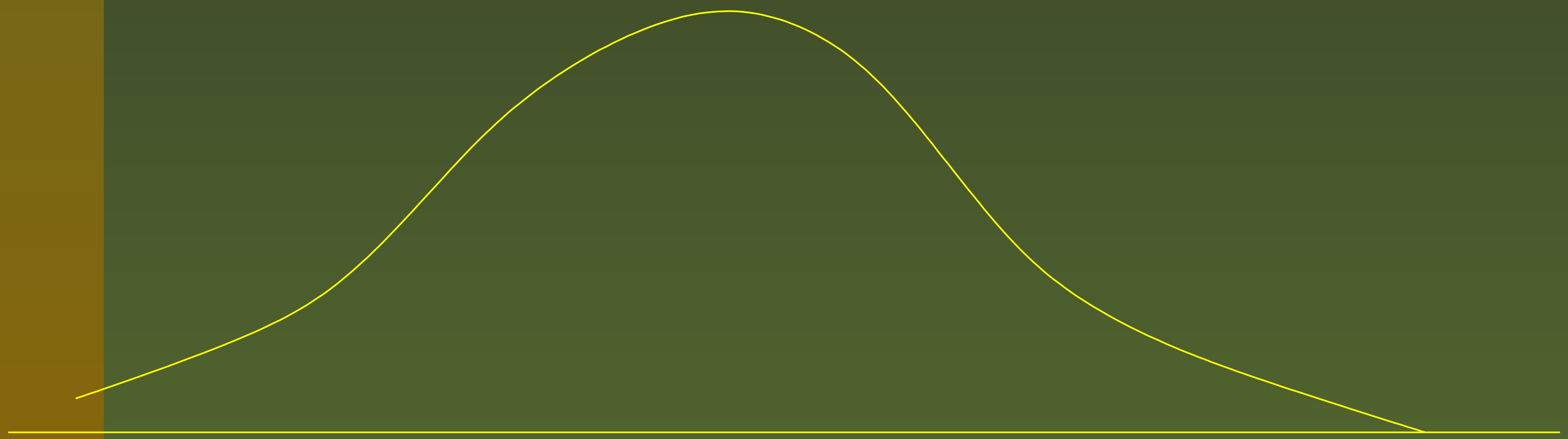
- If not, a general data structure for density functions should be found
- Computation difficulties may arise.

Discretisation

$$f : \Omega \to \mathbb{R} \quad \text{density function.}$$

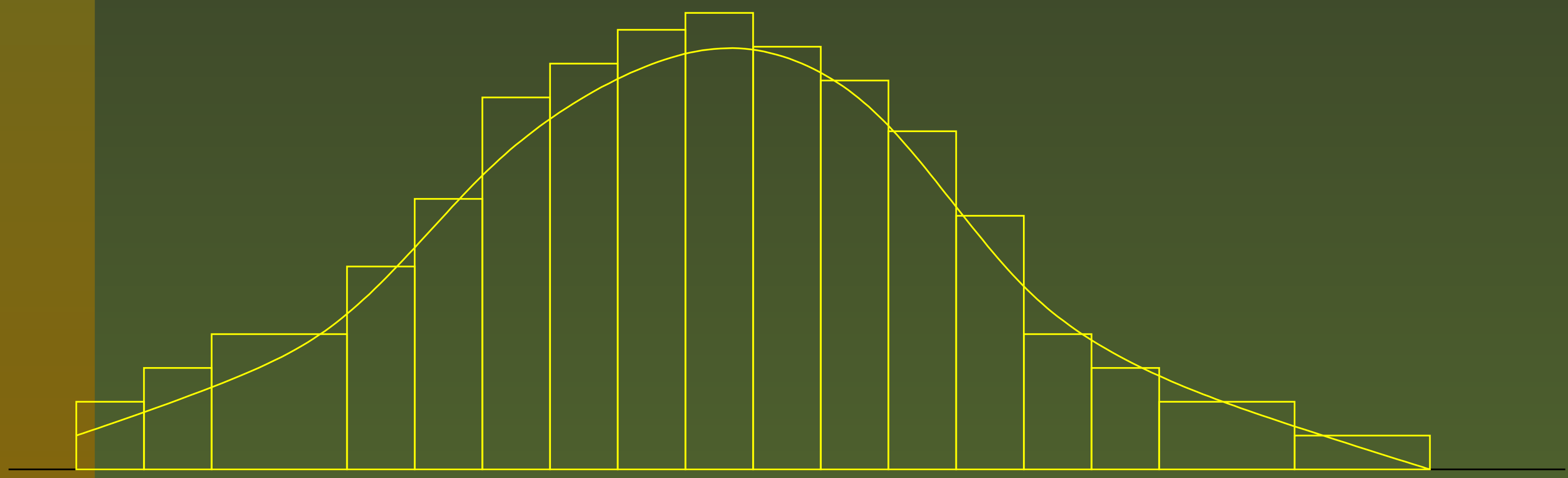


Discretisation

$$f : \Omega \to I\!R \quad \text{density function.}$$

$$\downarrow$$

$$f_D : \Omega \to I\!R \quad \text{histogram.}$$



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- It is an approximation.

Conditional Gaussian distributions

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- Restrictions on the network structure.

MTE potentials

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$$\phi(\mathbf{x}) = \phi(\mathbf{y}, \mathbf{z}) = a_0 + \sum_{i=1}^{m} a_i \exp\left\{ \sum_{j=1}^{d} b_i^{(j)} y_j + \sum_{k=1}^{c} b_i^{(d+k)} z_k \right\}$$

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ii. There is a partition $\Omega_1, \ldots, \Omega_k$ of $\Omega_{\mathbf{X}}$ such that $\phi(\mathbf{x}) = \phi_i(\mathbf{x}) \quad \text{if} \ \ \mathbf{x} \in \Omega_i$, where ϕ_i can be written as in i.

MTE potentials: example

$$\phi(x, z_1, z_2) = \begin{cases} 2 + e^{3z_1+z_2} & \text{if } \ 0 < z_1 \leq 1, \ 0 < z_2 < 2, \ x = 0 \\ 1 + e^{z_1+z_2} & \text{if } \ 0 < z_1 \leq 1, \ 2 \leq z_2 < 3, \ x = 0 \\ \frac{1}{4} + e^{2z_1+z_2} & \text{if } \ 1 < z_1 < 2, \ 0 < z_2 < 2, \ x = 0 \\ \frac{1}{2} + 5e^{z_1+2z_2} & \text{if } \ 1 < z_1 < 2, \ 2 \leq z_2 < 3, \ x = 0 \\ e^{z_1} + 2e^{z_2} & \text{if } \ 0 < z_1 \leq 1, \ x = 1 \\ 1 + e^{3z_1+z_2} & \text{if } \ 1 < z_1 < 2, \ x = 1 \end{cases}$$

MTE potentials: operations

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$$\phi^{\downarrow \mathbf{X}'}(\mathbf{y}', \mathbf{z}') = \sum_{\mathbf{y} \in \Omega_{\mathbf{Y} \setminus \mathbf{Y}'}} \left(\int_{\Omega_{\mathbf{Z}''}} \phi(\mathbf{y}, \mathbf{y}', \mathbf{z}, \mathbf{z}') d\mathbf{z}'' \right)$$

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- The class of MTE potentials is closed for the three operations.

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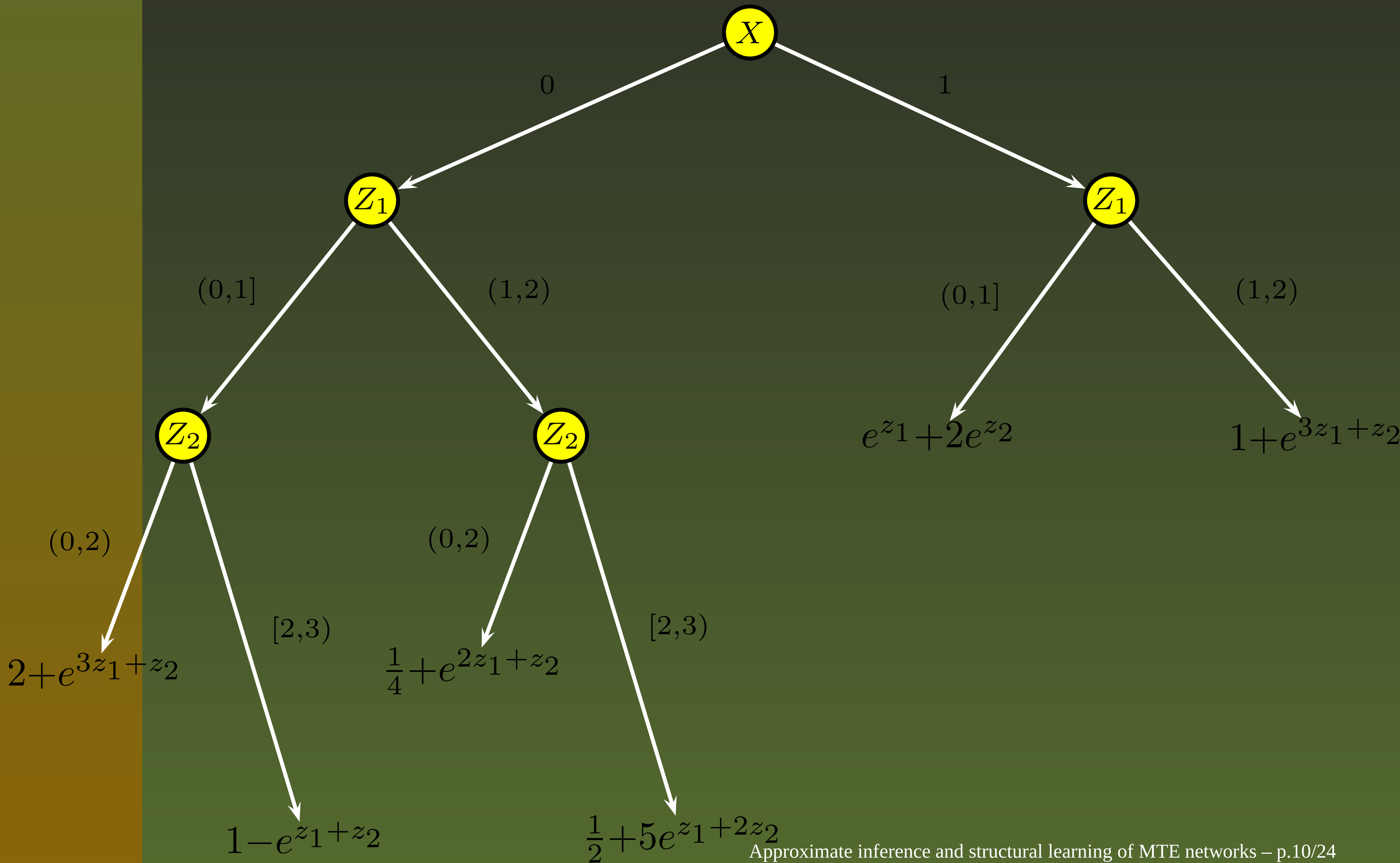
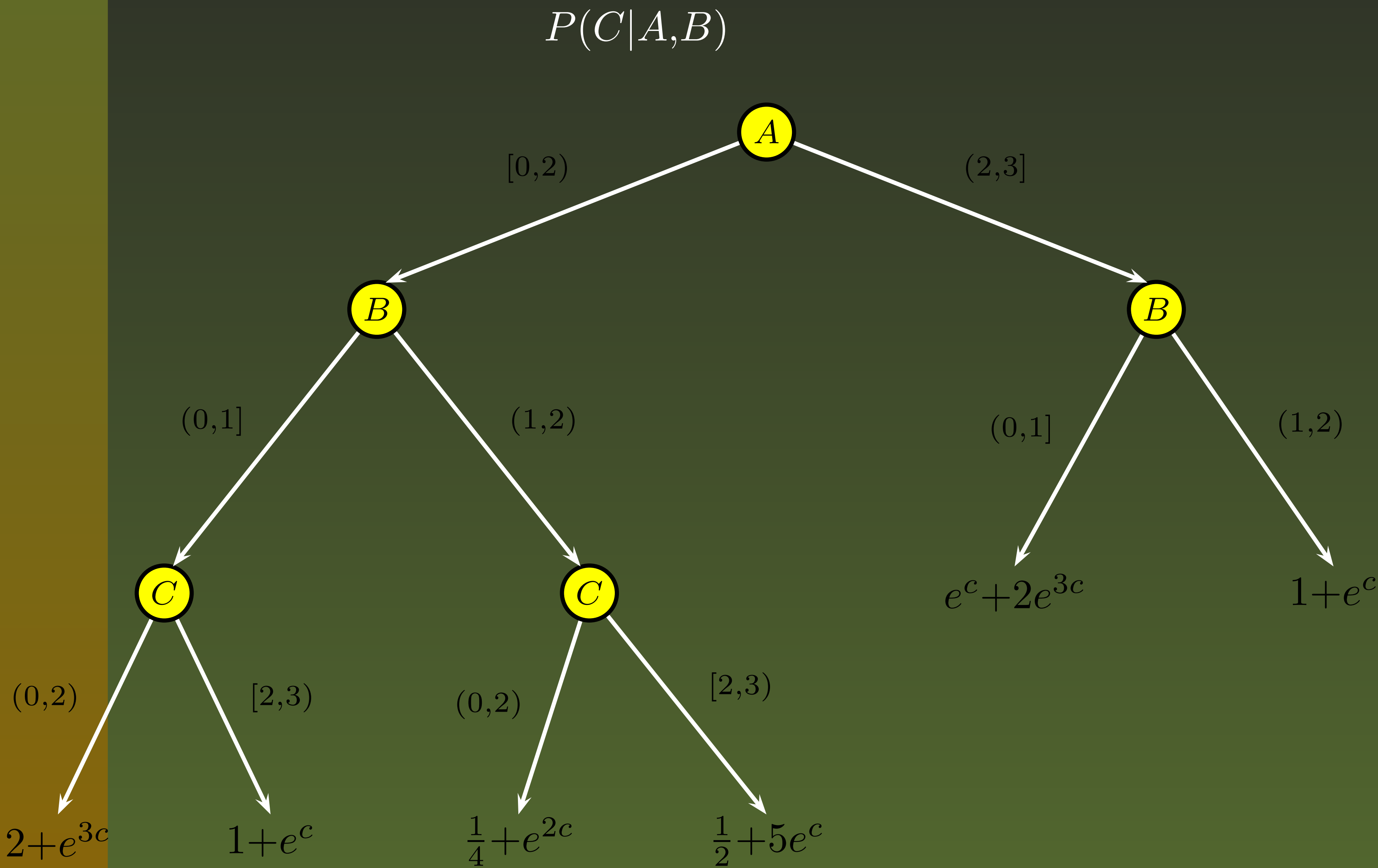
- An MTE potential f defined over $\mathbf{X}_1 \cup \mathbf{X}_2$ is a conditional MTE density if:
  - For each $\mathbf{x}_2 \in \Omega_{\mathbf{X}_2}$, $f^{R(\mathbf{X}_2=\mathbf{x}_2)}$ is an MTE density for $\mathbf{X}_1$.

A data structure for MTE densities



Example of conditional MTE density



Constructing the tree

- Fixed number of splits.

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- Variables selected accorging to their splitting gain:

$$SG(C_i) = \sum_i e_i log(e_i)$$

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- Goal: Find a Bayesian network G with variables $\mathbf{X}$, that agrees with the data D.

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- Measuring the accuracy of $(G, \hat{\theta})$.

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- After a movement, the affected conditional distributions are estimated.

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- The trees are pruned in order to reduce their size.

Reducing the size of a tree

- Distance between two trees:

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D(\mathcal{T},\mathcal{T}') &= E_{\phi_T}[(\phi_T - \phi_{T'})^2] \\
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- Approximate whenever $D(\mathcal{T},\mathcal{T}') < \epsilon$.

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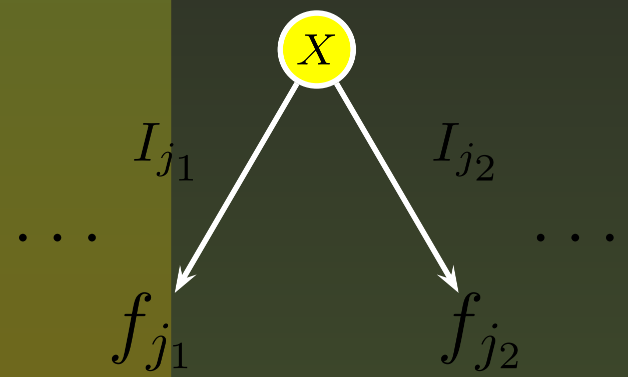
- Normalise f

Joining intervals



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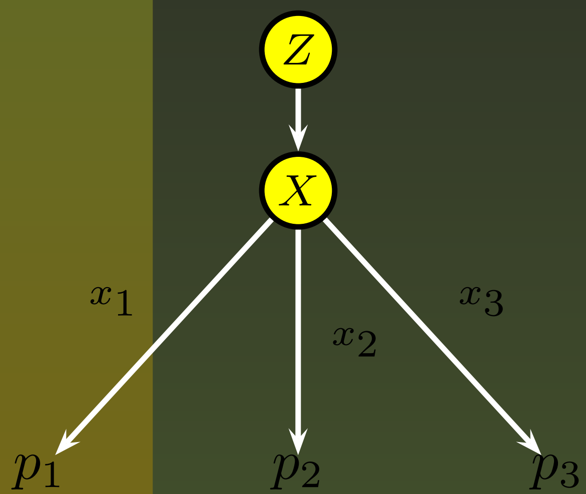
Pruning discrete variables



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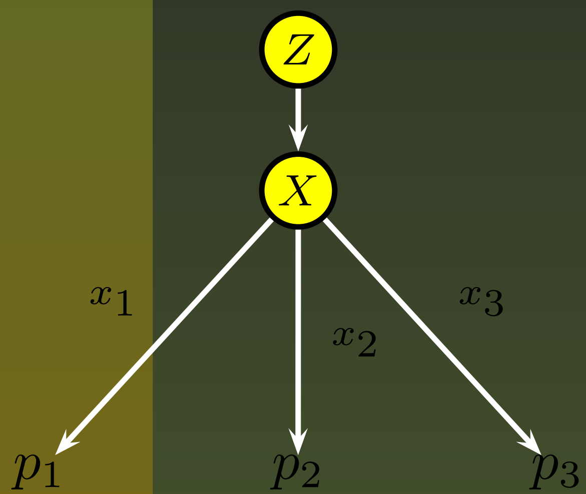

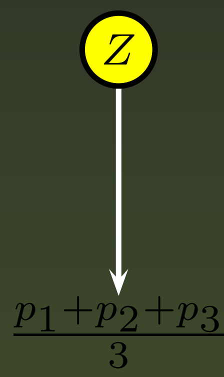

$$DK(p,q) = \sum_i p_i log(\frac{p_i}{q_i})$$

Pruning discrete variables

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$$\xi = -((0.5-\epsilon)log(0.5-\epsilon) + (0.5+\epsilon)log(0.5+\epsilon))$$

Prune if $DK(\phi, \overline{p}) < \xi$

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  - Estimate probabilities like $P(a < X < b)$.
  - Learn univariate densities from the sample.