Logical Approximations of Qualitative Probability

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Abstract

We provide approximations of qualitative probability, that is, comparative structures which are representable by probability measures. We introduce sequences of qualitative belief structures, based on the ideas of Depth-Bounded logics D'Agostino et al. (2013b), and identify the conditions under which:

1. a qualitative sequence *approximates* a qualitative probability;
2. a qualitative probability can be approximated.

Keywords: probability; uncertain reasoning; depth-bounded logics

1. Introduction and Motivation

Probability has long been acknowledged a key tool in AI research, and in combination with logic, has been put forward as a promising path to achieving explainable AI (see, e.g. Belle (2017)). However, work in Knowledge Representation and Reasoning has been traditionally vexed by the question: "where do the numbers come from?". The problem has led some researchers to considering qualitative approaches to uncertain reasoning, of which epistemic and nonmonotonic logics are well-known examples. Similarly, great attention has been devoted within the AI community to qualitative decision theory under uncertainty Dubois et al. (2003). The comprehensive survey Marquis et al. (2020) can be used as guide to the recent developments on both those distinct but related research threads.

The foundations of probability and statistics have a long tradition considering *qualitative probability* as a natural bridge between the logical and probabilistic representation of uncertainty. According to de Finetti (1951)

> [If representable by a probability measure, a qualitative probability] structure should be interpreted as an intermediate step between [algebraic logic] where the comparison is limited to the case of a pair of events such that one [logically implies] the other, and a quantitative theory where, owing to numerical evaluations, the comparison is fully specified.

This is certainly consonant with a slightly more *niche*, yet not unreasonable attitude taken in AI in response to the opening vexed question. For the comparison in probability of two events has often been seen as demanding 'less information' than its quantitative counterpart. In this spirit, the recent paper Delgrande et al. (2019) makes a case for knowledge-based systems to focus on qualitative probability.

Here is a rather subtle question which arises by taking an upfront logical perspective on the problem. For de Finetti's case for approximating probabilistic reasoning qualitatively relies on the mathematical fact that comparisons in probability are monotonic with respect to propositional logical consequence. And yet classical propositional logic does not provide an adequate model for expressing 'information'.

To see this in elementary terms, consider propositional variables p and q. Then

1. holding the information that $v(p \vee q) = 0$ is *sufficient* to holding the information that both
$$v(p) = 0 \text{ and } v(q) = 0.$$
2. however, holding the information that $v(p \vee q) = 1$ is *not sufficient* to holding either the information as to whether
$$v(p) = 1 \text{ or } v(q) = 1.$$

As a consequence of the duality between disjunction and conjunction, holding the information that a conjunction is true is sufficient to holding the information that both disjuncts are true. Finally, holding the information that $v(\neg p) = 1$ is sufficient to holding the information that $v(p) = 0$ (and conversely, of course).

We refer to the situation captured by 2. as a situation of *ignorance* – the agent just does not hold enough information to *decide* p. Essentially the same idea is referred to as *incomplete information* in Dubois et al. (1996), a notion defined as the situation in which relevant questions cannot be answered (in the context). What cannot be answered in 2. above is the question as to whether the agent holds the information concerning the truth value of p and q. We all have first-hand experience of this when prompted the (often annoying) message to the effect that "either your username or password is wrong". By exploiting this sense of ignorance, the website gives us just about the information we need to pay attention to the credentials we input.

Contrary to the received view then, classical propositional logic does allow us to represent some forms of ignorance, but it does not allow us to reason explicitly about it. Part of this problem is addressed successfully by the field of epistemic logic. This however is achieved at the price of spawning the problem of 'logical omniscience', which ultimately originates from the fact that (normal) epistemic logics *extend* classical logic.

An alternative approach consists in providing an *informational interpretation of classical logic*, as put forward by D'Agostino et al. (2013a); D'Agostino (2015). Building on that, the research reported in this note aims at putting forward a fully-fledged logical theory of approximated qualitative probability structures, and investigating the conditions under which those are quantitatively representable. In doing this, we will draw on a key model of approximate reasoning: the theory of Dempster-Shafer Belief Functions Shafer (1976, 1981). Hence we will pursue de Finetti's idea in a more general i.e. non additive, setting. Albeit unpalatable to our inspirator, this turns out to be the natural framework for our purpose. To see this intuitively, consider the following analogy. If ignorance is a disease, then acquiring information, can be a cure. But drugs first must be made available, second they must be paid for. So does information. Our setting can be viewed as a logical attempt to capture the idea that a larger budget may allow for a more effective way of producing drugs. And yet all budgets are limited by definition. Hence, to wrap up the analogy, approximating logical reasoning amounts to being able to reason sensibly in light of the limited information that we can invoke as a remedy to our ignorance.

This connects with the Dempster-Shafer framework as follows. Belief Functions quantify uncertainty by aggregating basic pieces of evidence, encoded in "probability mass assignments". This aggregation of evidence requires suitable *reasoning*, which is typically left implicit when framed set-theoretically, but is actually based on classical logic inference. Given the intractability of classical logic, the involved reasoning is actually far from trivial . The theory of depth-bounded boolean logics has been showed to provide a more fine-grained analysis of this process, separating the reasoning which just manipulates the initial evidence, from that which goes "beyond the evidence" Baldi and Hosni (2020).

Building on those results, we show that the hierarchy of depth-bounded boolean logics yield approximations of representable qualitative probability structures. This contributes to bridging the gap between the foundations of probability and statistics on the one hand, and the practical needs for more realistic reasoning (and decision making) under uncertainty in AI on the other.

2. Preliminaries

Our approach is logical, but no logical background exceeding this Section is necessary to follow our argument. Taking a logical approach to this subject means, among other things, identifying the probabilistic notion of event, with the elements of the set of *sentences* $\mathcal{SL}$ generated recursively from a countable propositional language $\mathcal{L}$, by means of the connectives in $\{\neg, \wedge, \vee\}$. As a consequence, the terms "event" and "sentence" will be used exchangeably in what follows. The constant $\bot$ stands for any contradiction. We will use lowercase Greek letters to refer to sentences, and uppercase Greek letters to refer to sets of sentences. Lowercase Latin from the final segment of the alphabet (and possibly with decorations) will be used to denote propositional variables in $\mathcal{L} = \{p_1, p_2, \ldots\}$. By construction of $\mathcal{SL}$, if $\psi = (\theta \wedge \varphi) \in \mathcal{SL}$ then both θ and φ belong to $\mathcal{SL}$. If needed, we call them *the immediate subsentences* of ψ (similarly, of course, for sentences involving negations and disjunctions). A propositional variable has no immediate subsentences. Then, the set of *subsentences* of φ is denoted by $S(\varphi)$ and is the smallest set closed under immediate subsentences – ditto for $S(\Gamma)$.

The informational view of propositional logic makes room to distinguish two *uses* of information by a reasoning agent. The first involves the information the agent actually holds. In the example of the previous Section, this amounts to the information that both subsentences in a disjunction are false, if the agent holds the information that the disjunction is false. This is referred to *actual information* in D'Agostino (2015). An informational reading of boolean tables gives us an immediate analogue for the actual information provided by a true conjunction. Finally, negation provides actual information about its immediate subsentence, whenever the information concerning its truth-value is held by the agent.

This suggests defining *zero-depth reasoning* as the closure under the actual information, i.e that provided by holding the information about the truth value of any sentence in $\mathcal{SL}$. Though this semantic intuition is useful to grasping the underlying idea of depth-bounded logics, for our present purposes, these are best introduced via derivability relations. To denote this notion of consequence we decorate the standard symbol for logical derivability. So we write

$$p \vdash_0 p \vee q \quad \text{and} \quad q \vdash_0 p \vee q,$$

to express zero-depth inferences granted by the use of actual information. Similarly, we have

$$p \wedge q \vdash_0 p \quad \text{and} \quad p \wedge q \vdash_0 q.$$

This motivates the definition of zero-depth consequence relations, which is given in general terms by referring to the set of rules collected in Table 1. These rules encode the valid principles for the manipulation of information actually

$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\ (\wedge\mathscr{I})$	$\dfrac{\neg\varphi}{\neg(\varphi \wedge \psi)}\ (\neg\wedge\mathscr{I}1)$
$\dfrac{\neg\psi}{\neg(\varphi \wedge \psi)}\ (\neg\wedge\mathscr{I}2)$	$\dfrac{\neg\varphi \quad \neg\psi}{\neg(\varphi \vee \psi)}\ (\neg\vee\mathscr{I})$
$\dfrac{\varphi}{\varphi \vee \psi}\ (\vee\mathscr{I}1)$	$\dfrac{\psi}{\varphi \vee \psi}\ (\vee\mathscr{I}2)$
$\dfrac{\varphi \quad \neg\varphi}{\bot}\ (\bot\mathscr{I})$	$\dfrac{\varphi}{\neg\neg\varphi}\ (\neg\neg\mathscr{I})$
$\dfrac{\varphi \vee \psi \quad \neg\varphi}{\psi}\ (\vee\mathscr{E}1)$	$\dfrac{\varphi \vee \psi \quad \neg\psi}{\varphi}\ (\vee\mathscr{E}2)$
$\dfrac{\neg(\varphi \vee \psi)}{\neg\varphi}\ (\neg\vee\mathscr{E}1)$	$\dfrac{\neg(\varphi \vee \psi)}{\neg\psi}\ (\neg\vee\mathscr{E}2)$
$\dfrac{\varphi \wedge \psi}{\varphi}\ (\wedge\mathscr{E}1)$	$\dfrac{\varphi \wedge \psi}{\psi}\ (\wedge\mathscr{E}2)$
$\dfrac{\neg(\varphi \wedge \psi) \quad \varphi}{\neg\psi}\ (\neg\wedge\mathscr{E}1)$	$\dfrac{\neg(\varphi \wedge \psi) \quad \psi}{\neg\varphi}\ (\neg\wedge\mathscr{E}2)$
$\dfrac{\neg\neg\varphi}{\varphi}\ (\neg\neg\mathscr{E})$	$\dfrac{\bot}{\varphi}\ (\bot\mathscr{E})$

Table 1: Introduction and Elimination rules

possessed by an agent, for each of the the connectives of the language. They are given in the format of INTroduction and ELIMination (INTELIM) rules for each connective, both when occurring positively (as the main connective of a formula) and negatively (in the scope of a negation), following general principles of natural deduction and tableaux systems. We refer to D'Agostino et al. (2013a); D'Agostino (2015) for further details and motivation.

Definition 1 $\Gamma \vdash_0 \varphi$ *if there is a sequence of sentences* $\varphi_1, \ldots, \varphi_m$ *such that* $\varphi_m = \varphi$ *and each* φ_i *is either in* Γ *or it is obtained by an application of the rules in Table 1 from sentences* φ_j *with* $j < i$.

With this definition in place we can say that an agent is ignorant about $\theta \in \mathscr{SL}$ if the agent cannot decide θ with zero-depth reasoning (possibly by using the premises in Γ).

Example 1 *Let* $\varphi = p \vee \neg p$. *Direct inspection of the rules in Table 1 shows that* φ *is not* 0*-depth derivable, i.e.* $\not\vdash_0 p \vee \neg p$.

But of course, logic does not end with manipulation of actual information. This is best illustrated with reasoning by cases, of which Savage's "Sure thing principle" is a well-known illustration.

> A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant. So, to clarify the matter to himself, he asks whether he would buy if he knew that the Democratic candidate were going to win, and decides that he would. Similarly, he considers whether he would buy if he knew that the Republican candidate were going to win, and again finds that he would. Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event obtains, or will obtain, as we would ordinarily say. Savage (1972)

The gist of the principle, lies in the use of *hypothetical information*. Savage's agent reaches a conclusion from information she does *not actually hold*, namely the actual winner of the next presidential elections. The decision is reached by drawing logical consequences in two mutually exclusive and jointly exhaustive "branchings", so to speak, of the evolution of the agent's actual information. In logic a similar pattern of inference is captured by the "elimination of disjunction" rule in natural deduction: infer ψ from the set of premisses $\{\theta \vee \varphi, \theta \to \psi, \varphi \to \psi.\}$. This rule has also a well-known counterpart in preferential non monotonic reasoning , where it is known as the *OR* rule: infer $\theta \vee \varphi \mid\!\sim \psi$ from $\theta \mid\!\sim \psi$ and $\varphi \mid\!\sim \psi$. The additional requirement in capturing the notion of hypothetical information[1] which lies at the core of depth-bounded boolean logics, is to the effect that the disjunctive premise features mutually exclusive disjuncts. This motivates the definition of *k-depth reasoning*.

Definition 2 *Let* $k > 0$. *Then* $\Gamma \vdash_k \varphi$ *if there is a* $\psi \in S(\Gamma \cup \{\varphi\})$ *such that* $\Gamma, \psi \vdash_{k-1} \varphi$ *and* $\Gamma, \neg\psi \vdash_{k-1} \varphi$.

Example 2 *Continuing Example 1, note that if we allow the agent to reason by cases on p, then it turns out that* both

$$p \vdash_0 p \vee \neg p \quad \textit{and} \quad \neg p \vdash_0 p \vee \neg p.$$

But by Definition 2, this is to say that $\vdash_1 p \vee \neg p$.

To further illustrate the idea of Definition 2, $k \in \mathbb{N}$ can be thought of as a "counter" which keeps track of how many instances of reasoning by cases are needed for the agent to decide a sentence of interest. In each of those steps, hypothetical information is *used as if it was* actual information, but for the agent to be able to do this coherently, they must keep track of those uses. This concurs to determining the cost of reasoning, which is formally measured in terms of the complexity of deciding a sentence at depth k. Results of D'Agostino, Gabbay and coauthors show that:

- $\vdash_0 \subset \vdash_1 \subset \cdots \subset \vdash_k \subset \cdots$, so the depth-bounded consequence relations form a hierarchy;

1. This is called *virtual information* in D'Agostino (2015)

- if $k \to \infty$ then $\vdash_k = \vdash$, i.e. in the limit, the hierarchy of depth-bounded boolean logics coincides with classical logic;
- for each k, $\vdash_k$ has a polynomial decision procedure.

Hence, the theory of depth-bounded boolean logics provides an ideal logical tool to achieve the goal of the investigation reported in this paper: For each $\vdash_k$ is a (tractable) approximation of classical logic (under the standard assumptions relating P to "feasible" computation). In addition, the larger the k, the better the approximation.

In the remainder of the paper we shall: define comparative probability structures based on depth-bounded boolean logics (Section 2.1); define approximations of qualitative probability structures (Section 3); show that probability functions are approximated by suitably defined hierarchy of belief functions (Section 4); identify the conditions under which approximate qualitative probability structures are asymptotically representable by a probability measure and, conversely, those conditions under which a representable qualitative probability structure can be approximated (Section 5).

2.1. Comparative Structures

A comparative structure is a pair $(\mathscr{A}, \preceq)$ where $\mathscr{A}$ is a boolean algebra and $\preceq$ is interpreted as a *qualitative probability (relation)* on $\mathscr{A}$. As usual, we assume that elements of $\mathscr{A}$ are closed under the boolean operations $\wedge, \vee, \neg$ (which we ambiguously denote with the symbols for logical connectives), whereas $\bot$ and $\top$ denote the top and bottom elements of the algebra, respectively. Recall that $\mathscr{A}$ has a natural lattice order associated to it, which is defined by $\theta \sqsubseteq \varphi$ iff $\theta \wedge \varphi = \theta$. Finally, we shall write $\theta \preceq \varphi$ to say that θ is no-more-probable-than φ, for any $\theta, \varphi \in \mathscr{A}$. The symmetric part of $\preceq$ is defined by $\theta \approx \varphi$ iff $[\theta \preceq \varphi$ and $\varphi \preceq \theta]$. The asymmetric part of $\preceq$ is defined by $\theta \prec \varphi$ iff $[\theta \preceq \varphi$ and it is not the case that $\theta \approx \varphi]$.

Definition 3 (Comparative structure) $(\mathscr{A}, \preceq)$ *is a* comparative structure *if*

1. $\preceq$ *is a total preorder over* $\mathscr{A}$*;*
2. $\bot \prec \top$*;*
3. *if* $\alpha \sqsubseteq \beta$ *then* $\alpha \preceq \beta$ *and*
4. *if* $\alpha \wedge \gamma = \bot$ *and* $\beta \wedge \gamma = \bot$ *then*

$$\alpha \preceq \beta \text{ if and only if } \alpha \vee \gamma \preceq \beta \vee \gamma.$$

This Definition is essentially due to de Finetti (1931) who introduced condition *4.* as the qualitative counterpart of additivity. As a consequence Definition 3 is often referred to as presenting the "de Finetti axioms". As recalled above, he thought of them as the logical core of uncertain reasoning, and conjectured that they would be necessary and sufficient for quantitative probabilistic reasoning. Take a finite set of events $\mathscr{A} \supseteq \Gamma = \{\gamma_1, \dots, \gamma_n\}$. Then any probability assignment $\gamma_i \mapsto p_i$, $i = 1 \dots, n$ leads to a relation $\preceq$ on Γ defined by

$$\gamma_i \preceq \gamma_j \text{ if } p_i \leq p_j,$$

which satisfies the de Finetti axioms. The converse, namely whether any relation $\preceq$ satisfying the de Finetti axioms is representable on the real-unit interval by a finitely additive measure, has been shown not to hold in 1959 by Kraft et al. (1959). Since then, a variety of paths have been followed to establishing (almost) representation, see Savage (1972); Kranz et al. (1971); Fishburn (1996). Indeed, establishing sufficiency turns out not to be a problem. For one effectively needs to find properties to impose on the order $\preceq$ which are stringent enough to determine a partition of equally likely events. When this happens, one can then quantify the probability of an event as the relative frequency of "favourable" cases over a sufficiently large number of equiprobable ones.

Definition 4 ((Almost) Representability) *A comparative structure* $(\mathscr{A}, \preceq)$ *is said to be :*

- representable *if there exists a unique finitely additive probability* P *such that* $\alpha \preceq \beta$ *iff* $P(\alpha) \leq P(\beta)$*;*
- almost representable *if there exists a unique finitely additive probability* P *such that* $\alpha \preceq \beta$ *implies* $P(\alpha) \leq P(\beta)$.

Note that the terminology adopted in the literature for the notion above is quite various: in particular the notion of *representable* comparative structure in some occurrences does not include uniqueness.

3. Approximations of Comparative Structures

3.1. Sequences of Forests

Let us begin by fixing some terminology and notation which is needed to formalise the idea of depth-bounded reasoning illustrated above.

Let F be any forest, whose vertices are sentences in $\mathscr{SL}$, and denote by $Le(F)$ the leaves of F. For any sentence $\gamma \in F$, we say that γ *k-decides* δ if $\gamma \vdash_k \delta$ or $\gamma \vdash_k \neg\delta$.

We say that a leaf $\alpha \in Le(F)$ is *locally closed* if α 0-decides δ, for each $\delta \in S(\alpha)$. A leaf which is not locally closed is said to be *locally open*. We say that a leaf $\alpha \in Le(F)$ is *globally closed* if $\alpha \vdash_0 \bot$ or α 0-decides any other leaf in F. A leaf which is not globally closed is said to be *globally open*.

Finally we say that a forest F is globally (locally) open, if each of its leaves is globally (locally) open. The same applies for globally or locally closed forests.

We will now define a sequence of k-depth forests, starting from an initial support $Supp \subseteq \mathscr{SL}$. The intended interpretation of *Supp* is that the sentences it contains represent the agent's *actual information*. It is convenient to assume that *Supp* is nonempty. So we need an extra symbol $*$, which is not part of the logical language, to denote the special case in which the agent holds no actual information, written $Supp = \{*\}$. We adopt the convention that $* \vdash_k \varphi$ stands for $\vdash_k \varphi$.

Depth-bounded reasoning then takes place as follows. Each open node is expanded by two new children nodes, representing an instance of reasoning by cases obtained by considering a certain piece of hypothetical information and its negation, respectively.

Definition 5 *For* $Supp \subseteq \mathscr{SL} \cup \{*\}$*, we define recursively,* a sequence $(F_k)_{k\in\mathbb{N}}$ of depth-bounded forests based on *Supp, as follows :*

1. *For* $k = 0$ *we let* F_0 *be a forest with no edges, and with the set of vertices equal to Supp. Clearly* $Le(F_0) = Supp$.
2. *The forest* F_k*, for* $k \geq 1$ *is obtained expanding at least one leaf* α *as follows:*
   - *If* α *is globally open, with two nodes* $\alpha \wedge \beta$ *and* $\alpha \wedge \neg\beta$*, where* β *is an undecided subsentence of some sentence in Supp, distinct from the root of* α.
   - *Otherwise, if* α *is globally closed, but locally open, with two nodes* $\alpha \wedge \beta$ *and* $\alpha \wedge \neg\beta$ *where* β *is an undecided subsentence of* α.
   - *Otherwise, if* α *is both locally and globally closed, with two nodes* $\alpha \wedge \beta$ *and* $\alpha \wedge \neg\beta$*, where* $\beta \in \mathscr{SL}$ *is a sentence whose variables do not already occur in* $Supp \cup \{\alpha\}$*, if there are any.*

Let us notice that, when $\mathscr{F}$ is defined over a language $\mathscr{SL}$ with finitely many propositional variables, the sequence of Depth-Bounded forest might be expanded only up to a certain F_k. More precisely, there will be some $k \in \mathbb{N}$, such that $F_n = F_k$ for each $n \geq k$.

3.2. Qualitative Belief and Plausibility Comparisons

Let $\Gamma \subseteq \mathscr{SL}$. With a useful abuse, we denote by $\mathscr{P}(\Gamma)$ both the subsets of Γ and the boolean algebra with domain $\mathscr{P}(\Gamma)$, with the usual set-operations.

Definition 6 *Let* $\Gamma \subseteq \mathscr{SL}$*. We call* Γ- qualitative mass *any* $\mathscr{M} = (\mathscr{P}(\Gamma), \preceq)$ *which is a comparative probability and satisfies:*

$$\text{For every } \varphi \in \Gamma, \text{ if } \varphi \vdash_0 \bot, \text{ then } \{\varphi\} \approx \emptyset$$

Definition 7 *For any* $\varphi \in \mathscr{SL}$*, and* Γ*-qualitative mass* $\mathscr{M}$ *the sets*

$$b_{\mathscr{M}}(\varphi) = \{\alpha \in \Gamma \mid \alpha \vdash_0 \varphi,\ \alpha \nvdash_0 \bot\}$$

and

$$pl_{\mathscr{M}}(\varphi) = \{\alpha \in \Gamma \mid \alpha \nvdash_0 \neg\varphi,\ \alpha \nvdash_0 \bot\}$$

are said to provide sufficient grounds *and* plausible grounds for φ*, respectively.*

Definition 8 (Qualitative belief and plausibility) *Let* $\mathscr{M} = (\mathscr{P}(\Gamma), \preceq)$ *be a* Γ*- qualitative mass structure.* The qualitative $\mathscr{M}$-based belief $\preceq^b$ *is defined by letting*

$$\varphi \preceq^b \psi \text{ if and only if } b_{\mathscr{M}}(\varphi) \preceq b_{\mathscr{M}}(\psi).$$

The qualitative $\mathscr{M}$-based plausibility $\preceq^{pl}$ *is defined by letting*

$$\varphi \preceq^{pl} \psi \text{ if and only if } pl_{\mathscr{M}}(\varphi) \preceq pl_{\mathscr{M}}(\psi).$$

3.3. Qualitative Sequences and their Properties

To complete our set up, we need to link the syntactical presentation of k-depth reasoning introduced in Subsection 3.1 to the qualitative version of belief and plausibility functions. For this we need some final bits of terminology.

Definition 9 (Qualitative sequence) *We say that* $\mathscr{F} = (\mathscr{F}_k)_{k\in\mathbb{N}}$ *is a* depth-bounded qualitative sequence *(just qualitative sequence for short), if* $(F_k)_{k\in\mathbb{N}}$ *is a sequence of depth-bounded forests, and each* $\mathscr{F}_k = (\mathscr{P}(Le(F_k)), \preceq_k)$ *is a* $Le(F_k)$*-qualitative mass.*

In what follows, we will denote by $\preceq_k^b$ and $\preceq_k^{pl}$ each of the qualitative $\mathscr{F}_k$-based belief and plausibility relations, respectively. We will also abbreviate $b_{\mathscr{F}_k}(\varphi)$ with $b_k(\varphi)$, for readability. Note that no further conditions is imposed at this stage on the various qualitative $\mathscr{F}_k$-based belief functions. In Section 5 we will illustrate the conditions under which the qualitative sequences determine in the limit a comparative structure, and in particular an almost representable one. Before doing that, let us flesh out some interesting properties of depth-bounded forests.

Definition 10 (Maximal forests) *Let* $(F_k)_{k\in\mathbb{N}}$ *be a sequence of depth-bounded forests and* $\Pi \subseteq \mathscr{SL}$*. We say that* a forest F_k is Π- maximal *if the number of sentences in* $Le(F_k)$ *which* 0*-depth decide* φ *for each* $\varphi \in \Pi$*, is maximal with respect to any other possible choice of hypothetical information at the given depth. We say that* the sequence is Π-maximal *if* F_k *is* Π*- maximal for each* $k \in \mathbb{N}$.

Our first result establishes that the structure of depth-bounded qualitative sequences provides the basis to approximate qualitative probability structures. More precisely, we first establish that the qualitative belief relations in Definition 8 satisfy weaker analogues of the properties of the (classical) qualitative belief relations introduced in Wong et al. (1991). The additivity axiom of Definition 3, in analogy with its quantitative counterpart, is not generally satisfied by qualitative belief relations. The axiom holds for our qualitative belief only when the leaves are locally closed, and the resulting relations essentially amount to comparative structures.

Lemma 11 *Let $\mathscr{F} = (\mathscr{F}_k)_{k \in \mathbb{N}}$ be a qualitative sequence. The relation $\preceq_k^b$ satisfies the following:*

1. *$\preceq_k^b$ is a total preorder.*
2. *$\bot \prec_k^b \top$.*
3. *For any $\varphi, \psi \in \mathscr{SL}$, if $\varphi \vdash_k \psi$ then there is a $n \geq k$ such that $\varphi \preceq_n^b \psi$. Moreover, if $\Pi = \{\varphi, \psi\}$ and F_k is Π - maximal, we get $\varphi \preceq_k^b \psi$ whenever $\varphi \vdash_k \psi$.*
4. *Let $\varphi, \psi, \gamma \in \mathscr{SL}$, with $\varphi \vdash \psi$ and $\psi, \chi \vdash_0 \bot$. Then there is a k such that $\varphi \vee \chi \prec_k^b \psi \vee \chi$*
5. *Let $\varphi, \chi, \psi \in \mathscr{SL}$, with $\varphi, \chi \vdash_0 \bot$,$\varphi \vdash_0 \psi$ and $\psi, \chi \vdash_0 \bot$. If $Le(F_k)$ is locally closed, then $\varphi \preceq_k^b \psi$ iff $\varphi \vee \chi \preceq_k^b \psi \vee \chi$.*

Proof

1. Follows immediately from the fact that $\preceq_k$ is a total order.
2. It follows from the fact that $\preceq_k$ is a qualitative mass, hence $b_k(\bot) \preceq_k \emptyset$ by Definition 6, and $\emptyset \prec_k b_k(\{\top\})$, by 2. in Definition 3.
3. If $\varphi \vdash_k \psi$, any n-depth forest F_n, for $n \geq k$, containing the virtual information used in a k-depth proof of ψ from φ, can be used for verifying the claim. Indeed, we will have that $\varphi, \alpha \vdash_0 \psi$, for every $\alpha \in Le(F_n)$. Hence, for every α such that $\alpha \vdash_0 \varphi$ (if there are any), we obtain $\alpha \vdash_0 \psi$. But this amounts to $b_n(\varphi) \subseteq b_n(\psi)$, hence we get $b_n(\varphi) \preceq_n b_n(\psi)$ and, by the definition of $\preceq_n^b$, $\varphi \preceq_n^b \psi$.

   For the second claim, from the assumption $\varphi \vdash_k \psi$, we know that there is a k-depth forest deriving ψ from φ. On the other hand, since F_k is maximal for $\{\varphi, \psi\}$, we will get $\varphi, \alpha \vdash_0 \psi$ for every $\alpha \in Le(F_k)$, and by the same reasoning as in the previous case, $\varphi \preceq_k^b \psi$.
4. Since $\varphi \vdash \psi$, there is a n such that $\varphi \vdash_n \psi$. Applying reasoning by cases, we can then obtain $\varphi \vee \chi \vdash_{n+1} \psi \vee \chi$. By 3. it then follows $\varphi \vee \chi \preceq_k^b \psi \vee \chi$, for some $k \geq n+1$.

   On the other hand, $b_k(\varphi) \cup b_k(\chi) \prec_k b_k(\varphi \vee \chi)$
5. Let us assume $\varphi, \psi, \chi \in \mathscr{SL}$ such that $\varphi, \chi \vdash_0 \bot$ and $\psi, \chi \vdash_0 \bot$. We will have that $\varphi \vee \chi \preceq_k^b \psi \vee \chi$ iff

   $$b_k(\varphi \vee \chi) \preceq_k b_k(\psi \vee \chi) \tag{1}$$

   Recall that, by the properties of $\vdash_0$, since $Le(F_k)$ is locally closed, we have that $\alpha \vdash_0 \varphi \vee \chi$ iff $\alpha \vdash_0 \varphi$ or $\alpha \vdash_0 \chi$, and $\alpha \vdash_0 \psi \vee \chi$ iff $\alpha \vdash_0 \psi$ or $\alpha \vdash_0 \chi$. Hence we get that $b_k(\varphi \vee \chi) = b_k(\varphi) \cup b_k(\chi)$ and $b_k(\psi \vee \chi) = b_k(\psi) \cup b_k(\chi)$. Moreover, by our initial assumption, $\alpha \nvdash_0 \varphi \wedge \chi$ and $\alpha \nvdash_0 \psi \wedge \chi$ for any $\alpha \in Le(F_k)$, hence we will have $b_k(\varphi) \cap b_k(\chi) = \emptyset$ and $b_k(\psi) \cap b_k(\chi) = \emptyset$. We thus have that (1) amounts to

   $$b_k(\varphi) \cup b_k(\chi) \preceq_k b_k(\psi) \cup b_k(\chi) \tag{2}$$

   which in turn, since $\preceq_k$ is a comparative structure, and in particular enjoys axiom 4 in Definition 3, entails:

   $$b_k(\varphi) \preceq_k b_k(\psi) \tag{3}$$

   hence $\varphi \preceq_k^b \psi$.

■

In preparation to the next results, we need some further notation and terminology. Let $\mathscr{F} = (F_k, \preceq_k)_{k \in \mathbb{N}}$ be a qualitative sequence. Given any $\Delta \subseteq Le(F_k)$, we denote by $d_{k'}(\Delta)$ the set of descendants of Δ, occurring in $Le(F_{k'})$, for any $k' \geq k$.

Definition 12 *A qualitative sequence $\mathscr{F} = (\mathscr{F}_k)_{k \in \mathbb{N}}$ is:*

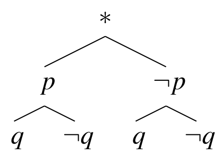
- Stable *if, for every $k \in \mathbb{N}$, and every $\Delta, \Gamma \subseteq Le(F_k)$, we have that $\Delta \preceq_k \Gamma$ implies $d_{k'}(\Delta) \preceq_{k'} d_{k'}(\Gamma)$ for every $k' \geq k$.*
- Refinable *if whenever $\alpha \preceq_k^b \beta$ for some $\alpha, \beta \in Le(F_k)$ and $k \in \mathbb{N}$, there is a $k' \geq k$ such that*

  $$\gamma \prec_{k'}^b d_{k'}(\{\alpha\}) \text{ for every } \gamma \in d_{k'}(\{\beta\}).$$
- Coverable *if whenever $\alpha \prec_k^b \beta$ for some $\alpha, \beta \in Le(F_k)$ and $k \in \mathbb{N}$, there is a $k' \geq k$ and $C \subseteq Le(F_{k'})$ such that $b_{k'}(C \cap (d_{k'}(\{\alpha\})) = \emptyset$ and*

  $$d_{k'}(\{\alpha\}) \cup C \approx_{k'}^b d_{k'}(\{b\}))$$

Stability is a key property for obtaining a comparative structure from qualitative sequences, but only in the limit. Let us illustrate with an example what happens in stable qualitative sequences.

Example 3 *Let $\mathscr{F} = (\mathscr{F}_k)$ be a qualitative sequence, with $Supp = \{*\}$. Assume that at depth 2, we have $\mathscr{F}_2 = (F_2, \preceq_2)$ with F_2 the tree :*



We have then $Le(F_1) = \{p, \neg p\}$ and $Le(F_2) = \{\neg p \wedge q, p \wedge \neg q, p \wedge q, \neg p \wedge \neg q\}$. Let us assume:

$$\{\neg p\} \preceq_1 \{p\} \tag{4}$$
$$\{\neg p \wedge q\} \preceq_2 \{p \wedge \neg q\} \preceq_2 \{p \wedge q\} \prec_2 \{\neg p \wedge \neg q\} \tag{5}$$

Now, by stability, we will also have

$$d_2(\neg p) = \{\neg p \wedge q, \neg p \wedge \neg q\} \preceq_2 d_2(p) = \{p \wedge \neg q, p \wedge q\}.$$

Let us now consider two formulas: $\neg p \vee q$ and $p \vee q$ and verify how they are ranked at depth 1 and 2. At depth one, we have $b_1(\neg p \vee q) = \{\neg p\}$ and $b_1(p \vee q) = \{p\}$, hence, by (4), we get $\neg p \vee q \preceq_1^b p \vee q$. On the other hand, we have: $b_2(\neg p \vee q) = \{\neg p \wedge q, p \wedge q, \neg p \wedge \neg q\}$ and $b_2(p \vee q) = \{\neg p \wedge q, p \wedge q, p \wedge \neg q\}$. Since $\{p \wedge \neg q\} \prec_2 \{\neg p \wedge \neg q\}$, we will have $b_2(p \vee q) \prec_2 b_2(\neg p \vee q)$, hence $p \vee q \prec_2^b \neg p \vee q$, which reverts the ordering at depth 1. On the other hand, it is easy to see that, for any formulas φ, ψ containing only the variables p, q we will have that $\varphi \preceq_n^b \psi$ for any $n \geq 2$, if and only if $\varphi \preceq_2^b \psi$.

We will now provide a Lemma, showing that the phenomenon in Example 3 generalizes to any stable sequence. This will be fundamental for obtaining a comparative structure in the limit in Lemma 18.

Lemma 13 *Let $\mathscr{F}$ be a stable qualitative sequence. For any $\varphi \in \mathscr{SL}$, there is a threshold $\tau(\varphi) \in \mathbb{N}$, such that $b_k(\varphi) = d_k(b_{\tau(\varphi)}(\varphi))$ for each $k \geq \tau(\varphi)$.*

Proof Pick $\tau(\varphi)$ to be the minimal number such that $\mathscr{F}$ is globally closed and φ is decided by each of the leaves in $Le(\tau(\varphi))$. We need to show that for $k \geq \tau(\varphi)$, the sentences in $Le(F_k)$ deriving φ can be seen as the union of the descendants, at depth k, of sentences deriving φ at depth $\tau(\varphi)$. Note that this is not always the case if we pick depth k less than $\tau(\varphi)$. Let $\beta \in b_k(\varphi)$, i.e. $\beta \in Le(F_k)$ and $\beta \vdash_0 \varphi$. Since $k \geq \tau(\varphi)$, is β is a descendant of a leaf a depth $\tau(\varphi)$, that is ,there is some $\alpha \in Lf(F_{\tau(\varphi)})$, such that $\beta \in d_k(\alpha)$. We want to show that such α is actually in $b_{\tau(\varphi)}(\varphi)$. Since $\beta \in d_k(\alpha)$, β is of the form $\alpha \wedge \chi$, and by definition of $\tau(\varphi)$, we can safely assume that χ does not contain any propositional variables occurring in φ. By definition of $\vdash_0$, we thus have that $\alpha \wedge \chi \vdash_0 \varphi$ implies $\alpha \vdash_0 \varphi$, hence $\alpha \in b_{\tau(\varphi)}(\varphi)$. ∎

Henceforth, for any two sentences φ, ψ, we will denote by $\tau(\varphi, \psi)$ the maximum of the thresholds $\tau(\varphi)$ and $\tau(\psi)$.

4. Approximating Probability Functions

This section prepares for the representation results of approximate qualitative probability structures, by building on recent results on the depth-bounded approximation of *probability* functions obtained in Baldi et al. (2020).

Recall that for $\Gamma \subseteq \mathscr{SL} \cup \{*\}$, $m\colon \Gamma \to [0,1]$ is a (quantitative) mass function over Γ if $\sum_{\alpha \in \Gamma} m(\alpha) = 1$ and $m(\alpha) = 0$ whenever $\alpha \vdash_0 \bot$. Unless otherwise stated, we will assume that $(F_k)_{k \in \mathbb{N}}$ is a sequence of depth-bounded forests based on *Supp*.

Definition 14 (Quantitative sequence) *We say that $\mathscr{F} = (F_k, m_k)_{k \in \mathbb{N}}$ is a depth-bounded quantitative sequence (quantitative sequence for short) if each m_k is a mass function over $Le(F_k)$, and for each $k > 0$:*

(i) $m_k(\gamma \wedge \alpha) + m_k(\gamma \wedge \neg\alpha) = m_{k-1}(\gamma)$ for any two leaves $\gamma \wedge \alpha$ and $\gamma \wedge \neg\alpha$ in $Le(F_k)$ with parent node $\gamma \in Le(F_{k-1})$;

(ii) $m_k(\gamma) = m_{k-1}(\gamma)$ if $\gamma \in Le(F_{k-1}) \cap Le(F_k)$.

Henceforth we will let $m(\Gamma) = \sum_{\alpha \in \Gamma} m(\alpha)$ and $m(\emptyset) = 0$, for $\Gamma \subseteq Supp$.

Definition 15 *Let $\mathscr{F} = (F_k, m_k)_{k \in \mathbb{N}}$ be a quantitative sequence. The k-depth belief function B_k and the k-depth plausibility function Pl_k are defined by letting:*

$$B_k(\varphi) = m_k(b_k(\varphi)) \text{ and } Pl_k(\varphi) = m_k(pl_k(\varphi)),$$

respectively.

In the following, we recall the key approximation result of Baldi et al. (2020), but we adapt it to the present setting, where we also admit an infinite language.

Theorem 16 *Let $P\colon \mathscr{SL} \to [0,1]$ be a finitely additive probability function. Then there is a quantitative sequence $\mathscr{F}$ based on $Supp = \{*\}$ such that, for each sentence φ, $P(\varphi) = \lim_{k \to \infty} B_k(\varphi)$.*

Proof First, let us consider the case where $|\mathscr{L}| = n$. Picking $Supp = \{*\}$ we define a quantitative sequence $\mathscr{F} = (F_k, m_k)_{k \in \mathbb{N}}$ based on $*$ such that $Le(F_n)$ is the set of maximal (classically) consistent conjunctions of literals from $\mathscr{L}$, denoted $At_{\mathscr{L}}$, and $m_n(\alpha) = P(\alpha)$ for each $\alpha \in At_{\mathscr{L}} = Le(F_n)$. Note that, once we fix m_n, Definition 14 forces us to uniquely determine all the m_k for $k < n$. Now, we obtain that, for each sentence $\varphi \in \mathscr{SL}$

$$P(\varphi) = \sum_{\substack{\alpha \in At_{\mathscr{L}} \\ \alpha \vdash \varphi}} P(\alpha) = \sum_{\substack{\alpha \in At_{\mathscr{L}} \\ \alpha \vdash_0 \varphi}} P(\alpha) = m_n(b_n(\varphi)) = B_n(\varphi).$$

Moreover, at depth n, all the propositional variables in $\mathscr{L}$ will have been used as hypothetical information, hence $F_k = F_n$ for any $k \geq n$, and $B_k(\varphi) = B_n(\varphi) = P(\varphi)$ for any $k \geq n$. This settles the claim.

If $\mathscr{L}$ is countable, a similar argument shows that for each sentence $\varphi \in \mathscr{SL}$ there is a $\tau(\varphi) \in \mathbb{N}$ such that $P(\varphi) = B_{\tau(\varphi)}(\varphi)$, and $B_n(\varphi) = B_{\tau(\varphi)}(\varphi)$ for each $n \geq \tau(\varphi)$. Hence, what is peculiar to the countable case in the fact that the index k at which $B_k(\varphi)$s equals $P(\varphi)$ may be distinct for distinct elements of $\mathscr{SL}$.

Whether $\mathscr{L}$ is finite or countable, $P(\varphi) = \lim_{k\to\infty} B_k(\varphi)$. ■

Note that the approximating quantitative sequence provided in the Theorem above is not unique, but different approximating measures can give rise to the same probability in the limit.

5. Representation Results

We are now ready to introduce the central results of this work, which identify the conditions of representation of approximate qualitative probability.

For $\mathscr{F} = (\mathscr{F}_k)_{k\in\mathbb{N}}$ a qualitative sequence, denote by $\mathscr{A}_\mathscr{F}$ the Lindenbaum-Tarski algebra over the language of $\mathscr{F}$. The elements of $\mathscr{A}_\mathscr{F}$, i.e. the equivalences of formulas in the language of $\mathscr{F}$ will be denoted by $\overline{\alpha}$, $\overline{\beta}$,...

Definition 17 (Limit structures) *We say that the qualitative structure* $(\mathscr{A}_\mathscr{F}, \preceq)$ *is* the limit of $\mathscr{F}$ *if* $\preceq$ *is defined by*

$$\overline{\alpha} \preceq \overline{\beta} \text{ iff there is a } k \text{ such that } \alpha \preceq^b_n \beta, \text{ for every } n \geq k,\ \alpha \in \overline{\alpha}, \text{ and } \beta \in \overline{\beta}.$$

Lemma 18 *If a qualitative sequence* $\mathscr{F}$ *is stable, then its limit* $(\mathscr{A}_\mathscr{F}, \preceq)$ *is a comparative structure.*

Proof The ordering property of $\preceq$ follow from Lemma 11. Reflexivity of $\preceq$ is easy. For transivity, assume $\overline{\alpha} \preceq \overline{\beta}$ and $\overline{\beta} \preceq \overline{\gamma}$. Then there exist $j,k \in \mathbb{N}$, such that $\alpha \preceq^b_n \beta$ and $\beta \preceq^b_m \gamma$ for every $n \geq k$, $m \geq j$, $\alpha \in \overline{\alpha}$, and $\beta \in \overline{\beta}$. Suppose w.l.o.g. that $k \geq j$. Then we get $\beta \preceq^b_n \gamma$ for every $n \geq k$, hence by the transitivity of $\preceq^b_n$, we get $\alpha \preceq^b_n \gamma$, for every $n \geq k$, and thus $\overline{\alpha} \preceq \overline{\gamma}$.

To see that $\preceq$ is total take $\overline{\varphi} \neq \overline{\psi}$. Now, since $\preceq^b_{\tau(\overline{\varphi},\overline{\psi})}$ is total, we will have either $\varphi \preceq^b_{\tau(\overline{\varphi},\overline{\psi})} \psi$ or $\varphi \preceq^b_{\tau(\overline{\varphi},\overline{\psi})} \psi$. Assuming w.l.o.g. that the first is the case, by Lemma 13, we will have $\varphi \preceq^b_n \psi$ for every $n \geq \tau(\overline{\varphi},\overline{\psi})$, hence $\overline{\varphi} \preceq \overline{\psi}$.

As for additivity, suppose that $\overline{\varphi \wedge \chi} = \bot$ and $\overline{\psi \wedge \chi} = \bot$. We will show that $\overline{\varphi} \preceq \overline{\psi}$ iff $\overline{\varphi \vee \gamma} \preceq \overline{\psi \vee \gamma}$. If $\overline{\varphi} \preceq \overline{\psi}$, by the definition of $\preceq$ there exists a k such that $\varphi \preceq_n \psi$ for every $n \geq k$, $\varphi \in \overline{\varphi}$, $\psi \in \overline{\psi}$. Now, pick a $k' \geq k$ such that $Le(F_{k'})$ is locally closed. Hence, by Lemma 11(5), $\varphi \preceq_{k'} \psi$ holds if and only if $\varphi \vee \chi \preceq_{k'} \psi \vee \chi$. Furthermore this holds for any $n \geq k'$. Hence we get $\overline{\varphi} \preceq \overline{\psi}$ iff $\overline{\varphi \vee \chi} \preceq \overline{\psi \vee \chi}$, as required.

Finally we show that, if $\overline{\varphi} \sqsubseteq \overline{\psi}$, then $\overline{\varphi} \preceq \overline{\psi}$. By the definition of the Lindenbaum-Tarski algebra, we will have that, for any $\varphi \in \overline{\varphi}$, $\psi \in \overline{\psi}$, $\varphi \vdash \psi$. On the other hand, since the depth-bounded logics approximate $\vdash$, there will be a k such that $\varphi \vdash_k \psi$. By Lemma 11(3) we will have that $\varphi \preceq^b_n \psi$, for $n \geq k$. Now, for any $n' \geq \max(n, \tau(\varphi,\psi))$, we will have $\varphi \preceq^b_{n'} \psi$, for any $\varphi \in \overline{\varphi}$, $\psi \in \overline{\psi}$. We have then obtained $\overline{\varphi} \preceq \overline{\psi}$.

■

Before introducing our first result, let us recall an important notion in Savage (1972).

Definition 19 *A comparative structure* $(\mathscr{A}, \preceq)$ *is said to be* fine *if, for any* $\alpha \in \mathscr{A}$ *such that* $\bot \prec \alpha$, *there exists a partition* $\beta_1, \ldots, \beta_n$ *of* $\mathscr{A}$ *such that* $\beta_i \prec \alpha$ *for each* $i = 1, \ldots, n$.

Note that fine algebras are necessarily infinite. In Savage (1972), it is shown that fine comparative structure are almost representable. This will be the key to our result in what follows.

Theorem 20 *If a qualitative sequence* $\mathscr{F}$ *is stable and refinable, then its limit* $(\mathscr{A}_\mathscr{F}, \preceq)$ *is almost representable.*

Proof By Lemma 18, we know that $(\mathscr{A}_\mathscr{F}, \preceq)$ is a comparative structure. As a consequence of the Savage's representation theorem Savage (1972), it suffices to show that $(\mathscr{A}_\mathscr{F}, \preceq)$ is *fine*. To see this, let $\overline{\varphi} \in \mathscr{A}_\mathscr{F}$ be such that $\bot \prec \overline{\varphi}$, i.e. there exists $\tau(\varphi) \in \mathbb{N}$ such that $\bot \prec^b_n \varphi$ for every $n \geq \tau(\varphi)$, $\varphi \in \overline{\varphi}$.

Now, pick any $\beta \in Le(F_{\tau(\varphi)})$. For any $\alpha \in b_{\tau(\varphi)}(\varphi)$ such that $\alpha \preceq^b_k \beta$, apply refinability, obtaining that, for some $k(\alpha,\beta) \in \mathbb{N}$, $\{\beta'\} \prec^b_{k(\alpha,\beta)} d_{k(\alpha,\beta)}(\alpha)$ for each $\beta' \in d_{k(\alpha,\beta)}(\beta)$.

Let now

$$k' = \max_{\beta \in Le(F_k)} \max_{\substack{\alpha \in b_{\tau(\varphi)}(\varphi) \\ \alpha \preceq^b_k \beta}} k(\alpha,\beta). \tag{6}$$

By stability, (6) yields $\beta' \prec^b_n d_n(\alpha)$ for every $\beta' \in Le(F_n)$, with $n \geq k'$, $\alpha \in b_{\tau(\varphi)}(\varphi)$.

Now, by Lemma 13 we have:

$$b_n(\varphi) = d_n(b_{\tau(\varphi)}(\varphi)) = \bigcup_{\alpha \in b_{\tau(\varphi)}(\varphi)} d_n(\alpha).$$

We thus get $\beta' \prec^b_n \varphi$, for every $\beta' \in \overline{\beta'}$, $\varphi \in \overline{\varphi}$ and $n \geq k'$, that is, $\overline{\beta'} \prec \overline{\varphi}$ for each $\overline{\beta'}$. Since β' ranges over all the leaves at depth n, it is easy to see that the corresponding $\overline{\beta'}$s form a partition of the boolean algebra $\mathscr{A}_\mathscr{F}$. This shows that the comparative structure $(\mathscr{A}_\mathscr{F}, \preceq)$ is fine, as required. ■

Note that, as a consequence of the result in Savage (1972) and Theorem 20, *refinability* forces the resulting limit structure to be infinite. We will now sketch a simple variant of our result, for the finite case.

Let us first recall the following definition from Krantz et al. (1971).

Definition 21 *We say that a comparative structure* $(\mathscr{A}, \preceq)$ *is equally spaced iff for any* $\varphi, \psi \in \mathscr{A}$ *such that* $\varphi \prec \psi$, *there exists a* $\gamma \in \mathscr{A}$ *such that* $\varphi \wedge \gamma = \bot$, *and* $\varphi \vee \gamma \approx \psi$.

In what follows, let us fix $\mathscr{F}$ to be a stable qualitative sequence, defined over a language $\mathscr{SL}$ with finitely many propositional variables. Recall that the sequence $\mathscr{F}$ reduces in this case to a finite sequence, say $\{\mathscr{F}_k\}_{k \in \{1,\ldots,n\}}$. Let us call $\mathscr{F}_n$ the *final* qualitative mass structure of $\mathscr{F}$. Note that, by the definition of depth-bounded forests, the support F_n of $\mathscr{F}_n$ will be locally closed. We obtain the representation result for the finite case as follows.

Theorem 22 *If* $\mathscr{F}$ *is a stable and coverable qualitative sequence over a finite language, its limit* $\mathscr{A}_{\mathscr{F}}$ *is representable.*

Proof Note that $\mathscr{A}_{\mathscr{F}}$ will be generated by finitely many propositional variables, hence it is finite. Theorem 6 in Krantz et al. (1971) shows that equally spaced finite comparative structures are representable. It will thus suffice to show that $\mathscr{A}_{\mathscr{F}}$ is equally spaced. Assume $\overline{\varphi} \prec \overline{\psi}$. Then for every $\varphi \in \overline{\varphi}, \psi \in \overline{\psi}$, we get $\varphi \prec_n^b \psi$. Since $\mathscr{F}$ is coverable and $\mathscr{F}_n$ is the final qualitative mass structure, we get that there is a set $C \subseteq Le(F_n)$ such that $b_n(C \cap \{\varphi\}) = \emptyset$ and $\{\varphi\} \cup C \approx_n^b \psi$. Let γ be the disjunction of the formulas in C. Since F_n is locally closed, we get $b_n(\{\varphi\} \cup C) = b_n(\varphi \vee \gamma)$. Hence we have $\varphi \vee \gamma \approx_n^b \psi$. On the other hand, note that $b_n(\alpha) = b_n(\alpha')$ for any $\alpha' \in \overline{\alpha}$. Hence, from $(\varphi \vee \gamma) \approx_n^b \psi$ we get $\overline{\varphi \vee \gamma} \approx \overline{\psi}$. This shows that $\mathscr{A}_{\mathscr{F}}$ is equally spaced. ∎

We conclude by showing that almost representable comparative structures can be qualitatively approximated. This makes crucial use of our result on the approximation of probability via Belief Functions (Theorem 16).

Theorem 23 *Let* $\mathscr{A}$ *be the Lindenbaum-Tarski algebra over* $\mathscr{L}$. *If* $(\mathscr{A}, \preceq)$ *is an almost representable comparative structure, then there exists a qualitative sequence* $\mathscr{F}$ *such that* $(\mathscr{A}, \preceq)$ *is the limit of* $\mathscr{F}$.

Proof Let P be the probability measure almost representing $(\mathscr{A}, \preceq)$. By Theorem 16, there is a sequence of k-depth belief functions B_k and k-depth mass functions approximating P. Let us define the corresponding $\preceq_k$ comparative structure, by letting $\Gamma \preceq_k \Delta$ iff $m_k(\Gamma) \leq m_k(\Delta)$ for each $\Gamma, \Delta \subseteq Le(F_k)$. For any $\alpha, \beta \in \mathscr{SL}$, we will then have $\alpha \preceq_k^b \beta$ iff $B_k(\alpha) \leq B_k(\beta)$. We are then left to prove that $\preceq$ is the limit of the $\preceq_k^b$, i.e. we need to prove that $\overline{\alpha} \preceq \overline{\beta}$ iff there exists a k such that $\alpha \preceq_n^b \beta$ for every $n \geq k$. Since P represents $\preceq$, from $\overline{\alpha} \preceq \overline{\beta}$ we get $P(\overline{\alpha}) \leq P(\overline{\beta})$. There exists then a B_k such that $B_k(\alpha) = P(\alpha)$ and $B_k(\beta) = P(\beta)$. Hence we will have $B_n(\alpha) \leq B_n(\beta)$ for every $n \geq k$, which by definition of $\preceq_n$, implies $\alpha \preceq_n^b \beta$. ∎

6. Conclusions and Future Work

We have presented a hierarchy of depth-bounded qualitative belief relations, which approximate classical comparative structures. We identified conditions to be imposed on such approximation sequences in order to obtain structures which are uniquely representable by classical, finitely additive, probability functions in the limit. This was only an initial step, towards the implementation of these bounded qualitative belief relations in concrete reasoning scenarios. The first future research direction is investigating the complexity of satisfiability and inference problems involving our qualitative approximations. In particular, we believe that the satisfiability problem will be tractable, since the results that we obtained in Baldi et al. (2020) should transfer rather smoothly to the qualitative setting.

Concerning inference, following a reviewer's suggestion, we plan also to investigate counterparts of credal and Bayesian networks, on the basis of both our quantitative approximations in Baldi et al. (2020) (recalled here in Section 4) and our qualitative approximations investigated here. The literature spanning from Wellman (1990), and still very active in artificial intelligence Mauá and Cozman (2020) is typically concerned with devising algorithms and assessing the complexity of inference problems related with the shape of the networks. Our line of work so far, on the other hand, has aimed at devising measures which are already tractable, even in the absence of assumptions of independence, as encoded in the networks. Understanding the relation between our work and various forms of qualitative networks in the literature, would require first a deeper investigation of the notion of conditional probability and independence in the qualitative bounded setting, that we have only partially developed for the quantitative case so far. In particular, for the qualitative setting, we plan to analyze the relation between conditioning and the use of hypothetical information in a more explicit form than what we did here. One possible route is considering comparative structures which take conditional object as primitives, on the model of what has been done already in early work on the subject, e.g. in Koopman (1940), and develop suitable approximations of those structures.

Finally, an essential part of our research program, both conceptually relevant and application-oriented, will be then to investigate the structures presented here in connection with decision-theoretic frameworks. In particular, we will develop a bounded notion of preference, on the model of the bounded qualitative belief presented here, and as a justification for its basic principles, as originally done in Savage (1972). Our general aim here is obtaining suitable representation theorems, providing principles of maximization of expected utility for bounded agents.

Acknowledgements

We are grateful to Marcello D'Agostino for very useful conversations on depth-bounded boolean logics. We also thank the reviewers for their careful reading of our manuscript and very insightful suggestions. The authors' research was partly funded by the Department of Philosophy "Piero Martinetti" of the University of Milan under the Project "Departments of Excellence 2018-2022" awarded by the Ministry of Education, University and Research (MIUR). Hosni also acknowledges funding from the Deutsche Forschungsgemeinschaft (DFG, grant LA 4093/3-1).

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