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Generalised Co-variation for Sensitivity Analysis in Bayesian Networks

Sensitivity analysis in Bayesian networks

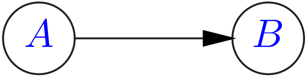
- Sensitivity analysis: a standard technique for studying effect of changes in model parameters on model output
- in Bayesian Networks: output probabilities are simple, multi-linear functions of network parameters (CPT entries)

Example:
A probability $\Pr(v)$ as a function of 2 network parameters x_1, x_2:

$$f_{\Pr(v)}(x_1, x_2) \;=\; c^{11} \cdot x_1 \cdot x_2 + c^{10} \cdot x_1 + c^{01} \cdot x_2 + c^{00}$$

- posterior $\rightarrow$ quotient
- assumption: (proportional) co-variation of other entries from same distribution

Co-variation in 1-way analysis



Varying a single parameter for a binary-valued variable:

	a_1	a_2
b_1	0.8	0.4
b_2	0.2	0.6

$\Longrightarrow$

	a_1	a_2
b_1	x	0.4
b_2	$1-x$	0.6

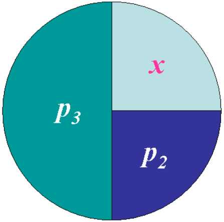
Varying a single parameter for a multi-valued variable:

	a_1	a_2
b_1	0.5	0.1
b_2	0.2	0.5
b_3	0.3	0.4

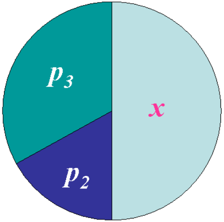
$\Longrightarrow$

<table>
<tr><th></th><th>a_1</th><th>a_2</th></tr>
<tr><td>b_1</td><td>x</td><td>0.1</td></tr>
<tr><td>b_2</td><td rowspan="2">$\left.\right\}\ 1-x$</td><td>0.5</td></tr>
<tr><td>b_3</td><td>0.4</td></tr>
</table>

Proportional co-variation



$$
\begin{aligned}
& \qquad\qquad\qquad\qquad\quad\; 1-x \\
p_2 &= 0.25 = 0.33 \cdot 0.75 \\
p_3 &= 0.50 = 0.67 \cdot 0.75
\end{aligned}
$$



$$
\begin{aligned}
& \qquad\qquad\qquad\qquad\quad\; 1-x \\
p_2 &= 0.17 = 0.33 \cdot 0.50 \\
p_3 &= 0.33 = 0.67 \cdot 0.50
\end{aligned}
$$

Motivation I

Why use proportional co-variation?

- standard approach
- assumed by sensitivity functions, algorithms & properties
- seems sensible
- works with any parameter
- optimal:

The CD-distance between the original distribution Pr and the new distribution $\Pr^*$

$$D(\Pr, \Pr^*) = \ln \max_w \frac{\Pr^*(w)}{\Pr(w)} - \ln \min_w \frac{\Pr^*(w)}{\Pr(w)}$$

is smallest under a proportional co-variation scheme [Chan & Darwiche, 2002]

- ...

Motivation II

Why use an alternative co-variation scheme?

- is standard most appropriate?
- do functions, algorithms & properties depend on scheme?
- is CD-distance really optimal?

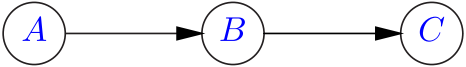
this was only proven for single parameter changes !

$n > 1$ simultaneous parameter changes can result in a smaller CD-distance [Chan & Darwiche, 2004]

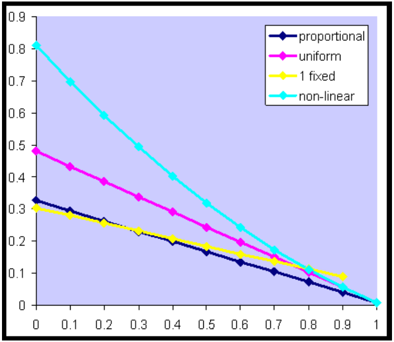
- again smallest under proportional co-variation ?
- this is unknown, and not obvious...

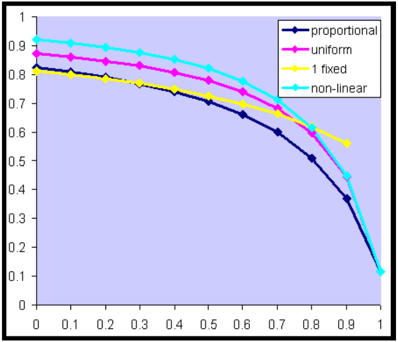
- who cares about CD-distance? ☺
- why *minimise 'disturbance'* in a sensitivity analysis?
- ...

Examples



Parameter $x = p(b_1 \mid a)$ with $x_0 = 0.2$ is varied in steps of 0.1(!)





Output of interest:
$\Pr(a, c)$ with $p_0 = 0.26$

Output of interest:
$\Pr(a \mid c)$ with $p_0 = 0.79$

Conclusions from examples

Consider a 1-way sensitivity function $f(x)$:

- alternatives to proportional co-variation don't necessarily ensure that $f(x_0) = p_0$
- $f(1)$, if defined, is independent of the co-variation scheme
- for linear $f(x)$: largest effect of alternative schemes is found for small values of x;
- for hyperbolic $f(x)$: this is not necessarily the case
- fixing[1] parameters preserves the standard form of $f(x)$, but constrains its domain

[1] (i.e. preventing a co-varying parameter from varying)

Conclusions and contributions

Consider a 1-way sensitivity function $f(x)$:

- the standard form of the sensitivity function is preserved for co-variation schemes linear in x:

> Pr($\mathbf{w}$) as a function of $x = \theta_{v_1|u}$ of t-valued variable V :
>
> $$f_{\mathbf{w}}(x) = (\alpha - \beta^{\gamma})\cdot x + (\beta^{\gamma} + \delta)$$
>
> $\alpha = \Pr(\mathbf{w}|v_1, u)\cdot\Pr(u)$, $\beta^{\gamma} = \sum_{k=2}^{t}\Pr(\mathbf{w}|v_k, u)\cdot\Pr(u)\cdot\gamma_{v_k|u}$, $\delta = \Pr(\mathbf{w}, \bar{u})$, and $\gamma_{v_k|u}\cdot(1-x)$, for all $\theta_{v_k|u}$, $1 < k \leq t$, is a co-variation scheme.

- this result extends to n-way functions
- constants can be computed with existing algorithms that construct and solve systems of equations.

Further contributions

Consider the CD-distance D:

- the generalised distance between old CPT $\Theta_{V|\mathbf{U}}$ and new CPT $\Theta^*_{V|\mathbf{U}}$
  (for each $\mathbf{u}_j$ we change parameter $\theta_{v_1|\mathbf{u}_j}$ and co-vary all other $\theta_{v_k|\mathbf{u}_j}$):

$$D_\gamma(\Theta_{V|\mathbf{U}}, \Theta^*_{V|\mathbf{U}}) = \ln \max_{\mathbf{u}_j} \left\{ \frac{\theta^*_{v_1|\mathbf{u}_j}}{\theta_{v_1|\mathbf{u}_j}}, \frac{1-\theta^*_{v_1|\mathbf{u}_j}}{\min_{k\neq 1} \gamma_{v_k|\mathbf{u}_j}{}^{-1} \cdot \theta_{v_k|\mathbf{u}_j}} \right\} - \ln \min_{\mathbf{u}_j} \left\{ \frac{\theta^*_{v_1|\mathbf{u}_j}}{\theta_{v_1|\mathbf{u}_j}}, \frac{1-\theta^*_{v_1|\mathbf{u}_j}}{\max_{k\neq 1} \gamma_{v_k|\mathbf{u}_j}{}^{-1} \cdot \theta_{v_k|\mathbf{u}_j}} \right\}$$

where $\gamma_{v_k|\mathbf{u}_j} \cdot (1-x)$ is a co-variation scheme.

- similar result for single parameter variation in $\Theta_{V|\mathbf{u}}$
- is this optimal for $\gamma =$ 'proportional' ?

Further contributions and future

Consider the CD-distance D:

- the lowerbound for n-way, single CPT co-variation under the proportional scheme:

$$D_p(\Theta_{V|\mathbf{U}}, \Theta^*_{V|\mathbf{U}}) \geq \max_{\mathbf{u}_j} \left| \ln \frac{\theta^*_{v_1|\mathbf{u}_j}}{\theta_{v_1|\mathbf{u}_j}} - \ln \frac{1-\theta^*_{v_1|\mathbf{u}_j}}{1-\theta_{v_1|\mathbf{u}_j}} \right|$$

- NB Chan (2005) introduced this expression as an approximation of the true distance
- What if we find an alternative co-variation scheme with

$$D_\gamma(\Theta_{V|\mathbf{U}}, \Theta^*_{V|\mathbf{U}}) \leq \max_{\mathbf{u}_j} \left| \ln \frac{\theta^*_{v_1|\mathbf{u}_j}}{\theta_{v_1|\mathbf{u}_j}} - \ln \frac{1-\theta^*_{v_1|\mathbf{u}_j}}{1-\theta_{v_1|\mathbf{u}_j}} \right| \quad ??$$